Richard Bird's Fan Club

Note Title 04/02/2008

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Richard Bird's Problem

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Consider a natural transformation tr :: L -> FL.
Suppose tr has the property that it returns an
F-structure of L-structures all of the same shape.
Consider the instance
         trG :: LG -> FLG.
I want to show that if trG is applied to an
L-sructure of G-structures all of the same shape,
then it returns a result that is an F-structure
of LG-structures all of the same shape.
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Solution: **Relation** algebra.

Pairs of Lists of Sums

([l], [r,l,r])

([l],[r,r,r]) ([l,r,l],[r,l,r])

([l],[l,l,l]) ([l,r,l], [l,r,l])

([l,l,l], [l,l,l])

Qu: Why **relations**?

Ans: "Theorems for free"

a,b types
f,g functions

$$tr \in L \to FL$$
$$tr \in \langle \forall a :: La \to FLa \rangle$$
$$tr \in \langle \forall f :: Lf \to FLf \rangle$$
i.e. $\langle \forall f,a : f \in a \to b : tr_b \circ Lf = FLf \circ tr_a \rangle$

$$ap \in \langle \forall a,b :: Ma \times M(a \to b) \to Mb \rangle$$
$$ap \in \langle \forall f,g :: Mf \times M(f \to g) \to Mg \rangle$$

relation on functions

Natural Transformation (Polymorphic Relation)

$$tr \in F \hookleftarrow G \equiv \langle \forall R :: FR \circ tr_a \supseteq GR \circ tr_b \rangle$$

Membership and Fan

$R \in a \leftarrow b$
Id identity relator

$$mem.F \in Id \hookleftarrow F$$

$$fan.F \in F \hookleftarrow Id$$

$$(mem.F)_a \setminus R = FR \circ (fan.F)_b$$

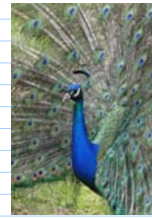
$$fan.F = mem.F \setminus Id$$

Examples: $FA = A \times A$, $GA = List\,A$, $HA = A + A$

$$fan.H = l \cup r$$

where l injects its argument into the left summand.
r injects its argument into the right summand.

$fan.G$
non-deterministicly constructs a list of arbitrary length with all elements equal.



The Fan Club

fgh	$(fan.F)GH$	$\circ\ (fan.G)H$	$\circ\ fan.H$
fhg	$(fan.F)GH$	$\circ\ G(fan.H)$	$\circ\ fan.G$
gfh	$F(fan.G)H$	$\circ\ (fan.F)H$	$\circ\ fan.H$
ghf	$F(fan.G)H$	$\circ\ F(fan.H)$	$\circ\ fan.F$
hfg	$FG(fan.H)$	$\circ\ (fan.F)G$	$\circ\ fan.G$
hgf	$FG(fan.H)$	$\circ\ F(fan.G)$	$\circ\ fan.F$

Examples: $FA = A \times A$, $GA = List\,A$, $HA = A + A$

gfh $\quad F(fan.G)H \circ (fan.F)H \circ fan.H$

$FGH \quad FH \quad H \quad I$ (types)

FG structure of (H structures of the same shape):

eg. $([l,l],[l,l,l]) \leftarrow\!\shortmid (l,l) \leftarrow\!\shortmid l \leftarrow\!\shortmid \bullet$

hfg $\quad FG(fan.H) \circ (fan.F)G \circ fan.G$

$FGH \quad FG \quad G \quad I$

FGH structure where G structures are all the same shape:

eg. $([l,r,l],[r,r,r]) \leftarrow\!\shortmid ([\bullet,\bullet,\bullet],[\bullet,\bullet,\bullet]) \leftarrow\!\shortmid [\bullet,\bullet,\bullet] \leftarrow\!\shortmid \bullet$

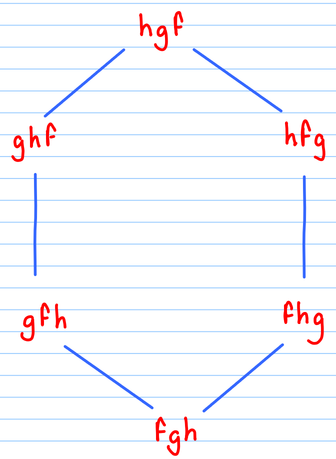
fhg $\quad (fan.F)GH \circ G(fan.H) \circ fan.G$

$FGH \quad GH \quad G \quad I$

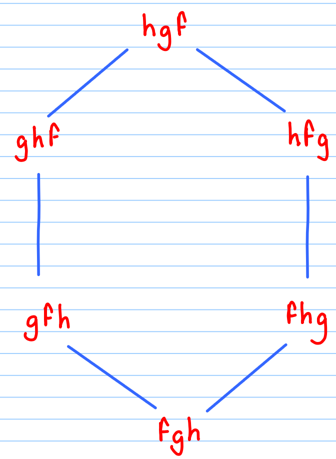
F structure of (GH structures of the same shape)

eg. $([l,r,l],[l,r,l]) \leftarrow\!\shortmid [l,r,l] \leftarrow\!\shortmid [\bullet,\bullet,\bullet] \leftarrow\!\shortmid \bullet$

Subset Ordering



Subset Ordering



$$
\begin{array}{ll}
 & hfg \\
= & \{\ \text{definition}\ \} \\
 & FG(fan.H) \ \circ\ (fan.F)G \ \circ\ fan.G \\
\supseteq & \{\ \ fan.F :: F \hookleftarrow Id\ , \\
 & \quad \text{i.e.}\ \ \langle \forall R :: FR \circ fan.F \supseteq fan.F \circ R \rangle \\
 & \quad R := G(fan.H)\ \ \} \\
 & (fan.F)GH \ \circ\ G(fan.H) \ \circ\ fan.G \\
= & \{\ \text{definition}\ \} \\
 & fhg
\end{array}
$$

Theorem $\quad gfh \cap hfg \subseteq fhg$.

In words, an FGH structure such that
all G structures have the same shape
and all GH structures have the same shape
has the property that
all H structures have the same shape.

$$gfh \cap hfg \subseteq fhg$$

$$
\begin{array}{l}
\Leftarrow \\
\quad \text{true}
\end{array}
$$

$$gfh \cap hfg \subseteq fhg$$

$\Leftarrow$ { $hfg = hf \circ fan.G$

$fhg = fh \circ fan.G$

modular identity }

$$gfh \circ (fan.G)^{\cup} \cap hf \subseteq fh$$

$\Leftarrow$

true

Simplification

$$R \circ S \cap T \subseteq R \circ U \quad \Leftarrow \quad S \cap R^{\cup} \circ T \subseteq U$$

$$\left\{ \begin{array}{l} R \subseteq fan \equiv mem \circ R \subseteq id \\ mem \circ fan \subseteq id \end{array} \right.$$

$$FR \circ fan \supseteq fan \circ R$$

$$mem \circ FR \supseteq R \circ mem$$

Complification

$$(fan.F)G = zip.F.G \circ G(fan.F)$$

September 1992 (STOP workshop) *Commuting Relators*
Roland Backhouse, Henk Doornbos, Paul Hoogendijk.
generic zip, fan (endorelators)

1996 *What is a datatype?* (Much later in JFP)
Paul Hoogendijk and Oege de Moor
generic membership

June 1997 (PhD thesis) *A Generic Theory of Datatypes*
Paul Hoogendijk
zip, membership, fan (relators of arbitrary arity)

Available from: www.cs.nott.ac.uk/~rcb/papers

LOOK

No gobbledeedook!